

# ON THE POSET OF VECTOR PARTITIONS

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**ABSTRACT.** We consider the poset of vector partitions of  $[n]$  into  $s$  components, denoted  $\Pi_{n,s}$ , which was first defined by Stanley in [10]. Sagan has shown in [9] that this poset is CL-shellable, and hence has the homotopy type of a wedge of spheres of dimension  $(n-2)$ . We extend on this result to show that  $\Pi_{n,s}$  is edge-lexicographic shellable. We then use this edge-labeling to find a recursive expression for the number of spheres, and show that when  $s=1$  the number of spheres is equal to the number of complete non-ambiguous trees, first defined in [1].

## 1. INTRODUCTION

For any positive integer  $n$ , we let  $[n]$  denote the set of integers  $\{1, 2, \dots, n\}$ . A *vector partition* of  $[n]$  into  $s+1$  components, or an  $(s+1)$ -*partition of  $[n]$* , is an  $(s+1)$ -tuple  $(P, w^1, \dots, w^s)$  where  $P$  is a partition of  $[n]$ , and each  $w^i$  is a labeling of the parts of  $P$  so that a part with  $\alpha$  elements is labeled by a subset of cardinality  $\alpha$ , and the labels form a partition of  $[n]$ . We define a poset  $\Pi_{n,s+1}$  with elements the  $(s+1)$ -partitions of  $[n]$  union a least element  $\hat{0}$  with relation  $(P, w^1, \dots, w^s) \leq (P', w^{1'}, \dots, w^{s'})$  if every part in  $P'$  is the union of parts in  $P$ , and the corresponding label sets in  $w^{i'}$  are the union of the label sets in  $w^i$  for each  $i$ . The poset of *vector partitions of  $[n]$  into  $s+1$  components*, or  $(s+1)$ -*partitions of  $[n]$* , denoted  $\bar{\Pi}_{n,s+1}$  is the induced subposet of  $\Pi_{n,s+1}$  with the least and greatest elements removed, i.e. it is the induced subposet with elements

$$\bar{\Pi}_{n,s+1} = \Pi_{n,s+1} - \{\hat{0}, ([n], \dots, [n])\}.$$

The relation  $P \leq P'$  where the parts of  $P'$  are the union of the parts in  $P$  forms the well known poset of partitions of  $[n]$ , denoted by  $\Pi_n$ .

The poset of vector partitions is an example of an exponential structure. Exponential structures were defined by Stanley in [10], where this particular type of exponential structure is mentioned as Example 2.5. In [9], Sagan studies the poset of vector partitions and shows that it admits a recursive atom ordering, which is equivalent to being CL-shellable (see [5, 7]).

The poset  $\Pi_{n,2}$  is a member of a family of posets that includes the poset of labeled subforests defined in [2]. We can see this by taking the partition  $P$  in an element  $(P, w^1) \in \Pi_{n,2}$  to be a partition of the vertices of  $K_n$ . In [2], Babson and Reiner define a poset in a similar manner to  $\Pi_{n,2}$ , however their underlying graph is any tree with  $n$  vertices rather than  $K_n$ . When the underlying graph is  $\text{Path}_n$  the order complex of this poset is the dual simplicial complex to the permutohedron.

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giving an edge labeling of a dual poset, which can possibly be used to find another expression for the number of spheres in  $\Delta(\Pi_{n,s})$ .

## 2. DEFINITIONS

This section contains a summary of the terminology used in this paper. We discuss simplicial complexes, poset theory, order complexes, and shelling orders. The reader is advised to see [8, Wachs, Poset Topology] for a complete introduction to this theory.

A *simplicial complex*  $\Delta$  with finite vertex set  $S$ , is a set of subsets of  $S$  such that

- $\{i\} \in \Delta$  for every  $i \in S$ , and
- if  $I \in \Delta$ , and  $J \subseteq I$ , then  $J \in \Delta$ .

Elements of a simplicial complex are called *faces*, and the dimension of a face  $F$  is equal to  $|F| - 1$ . Faces of dimension 0 are called *vertices*, faces of dimension 1 are called *edges*, and a *facet* is a face that is not properly contained in any other face. The dimension of a simplicial complex is the maximal dimension of all its faces. A simplicial complex is *pure* if all its facets are the same dimension.

A *poset* is a finite set  $P$  and a relation  $\leq$  on  $P$ , that satisfies:

- $x \leq x$  for all  $x \in P$ ,
- if  $x \leq y$  and  $y \leq x$ , then  $x = y$ ,
- if  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ .

A *length  $m$  chain* in  $P$  is a totally ordered subset  $\{x_0 < x_1 < \cdots < x_m\}$  of  $P$ . A *maximal chain* is a chain that is maximal with respect to inclusion. A poset  $S$  is an induced subposet of a poset  $P$  if  $S \subseteq P$ , and  $x \leq y$  in  $S$  if and only if  $x \leq y$  in  $P$ . For any  $x \leq y$  in  $P$ , the *interval*  $[x, y]$  is the induced subposet of  $P$  with elements all  $z \in P$  such that  $x \leq z \leq y$ . The *length* of an interval is the maximum of the length of all maximal chains in the interval. For a given subset  $S$  of  $P$ , an element  $x \in S$  is the *greatest* (respectively *least*) element of  $S$  if  $s \leq x$  (respectively  $x \leq s$ ) for all  $s \in S$ . The greatest element in a poset is denoted  $\hat{1}$ , and the least element is denoted  $\hat{0}$ . A poset is *bounded* if it contains both  $\hat{0}$  and  $\hat{1}$ . A poset  $P$  is *pure* if all maximal chains in  $P$  have the same length. A poset is *graded* if it is bounded and pure. If  $P$  is a bounded poset, the *proper part* of  $P$ , denoted by  $\overline{P}$ , is the induced subposet with elements  $P - \{\hat{0}, \hat{1}\}$ . If  $x$  and  $y$  are elements of a poset  $P$ , the notation  $x < y$  means that  $x < y$ , and there is no  $z \in P$  such that  $x < z < y$ . If  $x < y$ , then we say that  $x < y$  is an *edge* of  $P$ . An *atom* of a poset  $P$  with least element  $\hat{0}$  is an element  $a \in P$  such that  $\hat{0} < a$  is an edge of  $P$ . Posets  $P$  and  $Q$  are *isomorphic*, denoted  $P \cong Q$ , if there is a bijective map  $\phi : P \rightarrow Q$  such that  $\phi(x) \leq \phi(y)$  in  $Q$  if and only if  $x \leq y$  in  $P$ .

The *order complex* of a poset  $P$ , denoted  $\Delta(P)$ , is the simplicial complex with vertex set the elements of  $P$ , and with faces given by the chains of  $P$ .

It is clear from the definition that the poset  $\Pi_{n,s}$  is graded. In this paper, we will be studying topology of the order complex of the poset  $\overline{\Pi}_{n,s}$ . To do this we use a tool called shellability which is defined in the next paragraph.

If  $F$  is a face of a simplicial complex  $\Delta$ , we denote by  $\langle F \rangle$  the set of faces  $G \in \Delta$  such that  $G \subseteq F$ . A simplicial complex is *shellable* if there is a linear ordering  $F_1, F_2, \dots, F_k$  of its facets so that for all  $i \in \{2, \dots, k\}$ , the set  $\langle F_i \rangle \cap (\cup_{j=1}^{i-1} \langle F_j \rangle)$  is a pure  $(|F_i| - 2)$ -dimensional simplicial complex.

**Theorem 2.1** (Björner and Wachs [5]). *A shellable simplicial complex has the homotopy type of a wedge of spheres, where for each  $m$ , then number of  $m$ -dimensional spheres is the number of  $m$ -dimensional facets  $F_i$  whose boundary is contained in  $\cup_{j=1}^{i-1} \langle F_j \rangle$ . Such facets are called homology facets.*

There are well known techniques which can be used to show that the order complex of a graded poset is shellable. The topology of the order complex of a graded poset is not interesting since it is a contractible space. However, a shelling of the order complex of a bounded poset induces a shelling on the order complex of the proper part of this poset, as described in Proposition 2.2 below. In this paper we use a technique called EL-shellability, which we introduce in this section, to show that  $\Delta(\Pi_{n,s})$  and hence  $\Delta(\overline{\Pi}_{n,s})$  is shellable.

**Proposition 2.2.** *Suppose that  $P$  is a bounded poset, and that  $C_1, C_2, \dots, C_t$  is an ordering of the maximal chains of  $P$  that gives a shelling of the order complex  $\Delta(P)$ . Then  $\overline{C}_1, \overline{C}_2, \dots, \overline{C}_t$  is an ordering of the maximal chains of  $\overline{P}$  that gives a shelling of  $\Delta(\overline{P})$ . Here  $\overline{C}_i$  is the totally ordered subset  $C_i - \{\hat{0}, \hat{1}\}$ .*

We will now describe EL-shellability, a method developed by Björner and Wachs in [3, 4, 5, 6, 7] to show that the order complex of a bounded poset is shellable. If  $P$  is a bounded poset, an *edge labeling* of  $P$  is a map  $\lambda : \mathcal{C}(P) \rightarrow \Lambda$ , where  $\mathcal{C}(P)$  is the set of edges of  $P$ , and  $\Lambda$  is a totally ordered set. If  $P$  has an edge labeling  $\lambda$ , and  $C = x_1 < \dots < x_m$  is a chain of  $P$ , then we call the word  $\lambda(C) = \lambda(x_1 < x_2)\lambda(x_2 < x_3) \dots \lambda(x_{m-1} < x_m)$  the *chain label* of the chain  $C$ . We say that a chain  $C$  is *increasing* if  $\lambda(C)$  is strictly increasing, and that  $C$  is *decreasing* if  $\lambda(C)$  weakly decreasing.

Suppose that  $P$  is a bounded poset. An *edge-lexicographical labeling* or *EL-labeling* of  $P$  is an edge labeling such that for each interval  $[x, y]$  in  $P$ , there is a unique increasing maximal chain, which lexicographically precedes the other maximal chains of  $[x, y]$ . A bounded poset that admits an EL-labeling is said to be *edge-lexicographic shellable* or *EL-shellable*.

**Theorem 2.3** (Björner [3], Björner and Wachs [5]). *Suppose  $P$  is a bounded poset with an edge lexicographical labeling. Then the lexicographical ordering of the maximal chains of  $P$  is a shelling of  $\Delta(P)$ . The induced order on the maximal chains of  $\overline{P}$  is a shelling of  $\Delta(\overline{P})$ , where the number of  $i$ -spheres in  $\Delta(\overline{P})$  is equal to the number of decreasing maximal chains of length  $i + 2$  in  $P$ .*

### 3. AN EL-LABELING OF $\Pi_{n,s+1}$

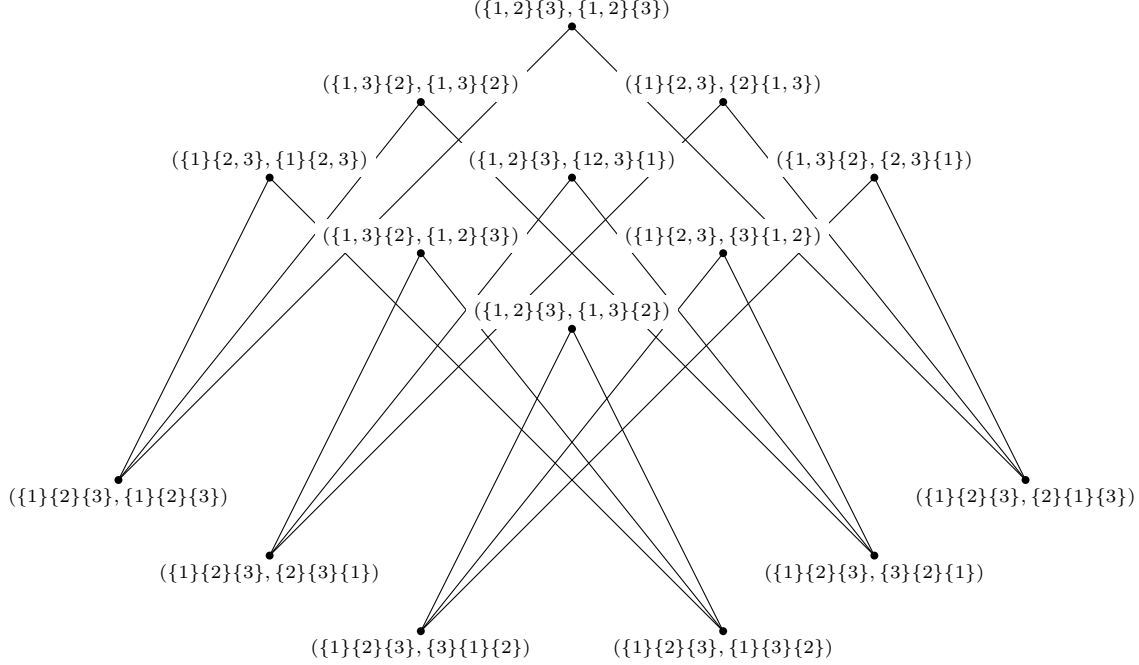
Suppose  $(P, w^1, \dots, w^s)$  is a vertex of  $\Pi_{n,s+1} - \{\hat{0}\}$ . We call  $w^i$  the  $i$ th labeling of  $P$ , and if  $I$  is a part in  $P$ , we call its corresponding label in  $w^i$  the  $i$ th label of  $I$ . From now on, we will list the parts of  $P$  in the order of their minimal element (a part  $I$  is listed earlier than a part  $J$  if the lowest

integer in  $I$  is lower than the lowest integer in  $J$ ), and we will represent each  $w^i$  by ordering the labelings by the parts they label in  $P$ . For example

$$(P, w^1, w^2) = (\{1, 4, 6\}\{2, 3\}\{5\}, \{2, 3, 5\}\{4, 6\}\{1\}, \{3, 5, 6\}\{2, 4\}\{1\})$$

means that the part  $\{1, 4, 6\}$  has 1st label  $\{2, 3, 5\}$  and 2nd label  $\{3, 5, 6\}$ , the part  $\{2, 3\}$  has 1st label  $\{4, 6\}$  and 2nd label  $\{2, 4\}$ , and the part  $\{5\}$  has 1st label  $\{1\}$  and 2nd label  $\{1\}$ .

**Example 3.1.** We depict the Hasse diagram of the poset  $\overline{\Pi}_{3,2}$ .



Recall that  $\Pi_n$  denotes the poset of partitions of  $[n]$ , whose elements are partitions of  $[n]$ , with relation  $x \leq y$  if and only if every part in  $y$  is the union of parts in  $x$ . We will now describe an EL-labeling  $\chi$  of the poset  $\Pi_n$ , which is defined by Wachs in [11].

**Theorem 3.2** (Wachs [11]). Suppose  $x \leq y$  in  $\Pi_n$ , and that  $y$  is obtained from  $x$  by merging two parts  $B_1$  and  $B_2$ . Let  $\chi : \mathcal{C}(\Pi_n) \rightarrow \mathbb{Z}$  be defined as

$$\chi(x \leq y) = \max(B_1 \cup B_2).$$

The map  $\chi$  defines an EL-labeling of the poset  $\Pi_n$ .

**Proposition 3.3.** Any interval  $[(P, w^1, \dots, w^s), (P'', w^{1''}, \dots, w^{s''})]$  in  $\Pi_{n,s+1}$  (where  $(P, w^1, \dots, w^s) \neq \hat{0}$ ) is isomorphic to the interval  $[P, P'']$  in the poset  $\Pi_n$ .

*Proof.* It is clear by the definition of  $\Pi_{n,s+1}$ , that the map

$$(P', w^{1'}, \dots, w^{s'}) \mapsto P'$$

defines an isomorphism of posets

$$[(P, w^1, \dots, w^s), (P'', w^{1''}, \dots, w^{s''})] \cong [P, P''].$$

□

Henceforth, we will identify atoms

$$(\{1\}\{2\}\dots\{n\}, \{w_1^1\}\dots\{w_n^1\}, \dots, \{w_1^s\}\dots\{w_n^s\})$$

in  $\Pi_{n,s+1}$  with the word

$$w_1^1 \dots w_n^1 \dots w_1^s \dots w_n^s.$$

With this identification, the atoms can be ordered by the lexicographical ordering on  $[n]^{ns}$ . For each element  $(P, w^1, \dots, w^s) \in \Pi_{n,s+1} - \{\hat{0}\}$ , there is a unique atom  $w_1^1 \dots w_n^1 \dots w_1^s \dots w_n^s$  of  $\Pi_{n,s+1}$  which is less than  $(P, w^1, \dots, w^s)$ , and is earliest in the lexicographic order on the atoms. We call  $w_1^1 \dots w_n^1 \dots w_1^s \dots w_n^s$  the *atom of*  $(P, w^1, \dots, w^s)$ , and denote it by  $A(P, w^1, \dots, w^s)$ . If  $(P, w^1, \dots, w^s) \in \Pi_{n,s+1}$ , we denote by  $w_j^i$  the  $(n(i-1) + j)$ th element of  $A(P, w^1, \dots, w^s)$ . The atom of  $(P, w^1, \dots, w^s)$  can be obtained as follows: Suppose that  $\{k_1, \dots, k_p\}$  is a part in  $P$  with  $i$ th label  $\{j_1, \dots, j_p\}$ , where  $k_1 < \dots < k_p$  and  $j_1 < \dots < j_p$ . Then  $w_{k_\alpha}^i = j_\alpha$  for all  $\alpha \in \{1, \dots, p\}$ . For example

$$(\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}, \{2\}\{1\}\{5\}\{3\}\{4\}\{8\}\{6\}\{7\})$$

is the atom of

$$(\{1, 4, 8\}\{2, 3, 7\}\{5, 6\}, \{2, 3, 7\}\{1, 5, 6\}\{4, 8\})$$

and it is identified with the word

$$21534867.$$

We will now define an edge labeling  $\lambda : \mathcal{C}(\Pi_{n,s+1}) \rightarrow \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$  of  $\Pi_{n,s+1}$ , where the ordering on  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$  is the lexicographical ordering. This labeling is similar to the labelings defined in [11] and [12], in that it labels the lowest edges with “some version” of 0, and labels some other edges with “some version” of a labeling of the partition lattice, in this case Wachs’ label in Theorem 3.2:

- Suppose

$$(P, w^1, \dots, w^s) \triangleleft (P', w^{1'}, \dots, w^{s'}),$$

and

$$A(P, w^1, \dots, w^s) \neq A(P', w^{1'}, \dots, w^{s'}).$$

Then

$$\lambda((P, w^1, \dots, w^s) \triangleleft (P', w^{1'}, \dots, w^{s'})) = (k, i, j)$$

where  $(k, i, j)$  is the lexicographically first element of  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$  such that  $w_k^i \neq w_k^{i'}$  and  $w_k^{i'} = j$ .

- Suppose

$$(P, w^1, \dots, w^s) \triangleleft (P', w^{1'}, \dots, w^{s'}),$$

and

$$A(P, w^1, \dots, w^s) = A(P', w^{1'}, \dots, w^{s'}).$$

Then

$$\lambda((P, w^1, \dots, w^s) \triangleleft (P', w^{1'}, \dots, w^{s'})) = (n, \max(I \cup J), 0),$$

where  $I$  and  $J$  are the parts in  $P$  that are merged to give  $P'$ .

- For any edge

$$\hat{0} \leq a,$$

where  $a$  is an atom of  $\Pi_{n,s+1}$ ,

$$\lambda(\hat{0} \leq a) = (n-1, s+m, 0)$$

where  $a$  is the  $m$ th atom in the lexicographical ordering on the atoms of  $\Pi_{n,s+1}$ .

We make the following observations about this edge labeling before proceeding with the proof of Theorem 3.5, which shows that  $\lambda$  is an EL-labeling.

**Lemma 3.4.** *The edge labeling  $\lambda$  satisfies the following five conditions:*

- (1) *If  $(P, w^1, \dots, w^s) \leq (P', w^{1'}, \dots, w^{s'})$  in  $\Pi_{n,s+1}$ , then  $A(P', w^{1'}, \dots, w^{s'}) \leq A(P, w^1, \dots, w^s)$  in the lexicographical ordering on atoms.*
- (2) *Any chain  $\hat{0} \leq C_1 \leq \dots \leq C_d$  that contains an edge  $x \leq y$  such that  $A(x) \neq A(y)$ , is not increasing.*
- (3) *If  $(P, w^1, \dots, w^s) \leq (P', w^{1'}, \dots, w^{s'})$  is an edge such that  $A(P, w^1, \dots, w^s) \neq A(P, w^{s'}, \dots, w^{s'})$  and  $\lambda((P, w^1, \dots, w^s) \leq (P', w^{1'}, \dots, w^{s'})) = (k, i, j)$ , then  $w_k^i > w_k^{i'} = j$ .*
- (4) *If  $(P, w^1, \dots, w^s) \leq (P', w^{1'}, \dots, w^{s'})$  is an edge such that  $A(P, w^1, \dots, w^s) \neq A(P, w^{s'}, \dots, w^{s'})$  and  $\lambda((P, w^1, \dots, w^s) \leq (P', w^{1'}, \dots, w^{s'})) = (k, i, j)$ , then  $P'$  is obtained from  $P$  by merging the part that contains  $k$  with the part whose  $i$ th label contains  $j$ .*
- (5) *Suppose  $I = [(P, w^1, \dots, w^s), (P', w^{1'}, \dots, w^{s'})]$  is an interval in  $\Pi_{n,s+1}$ , and that  $(k, i, j)$  is the first triple such that  $w_k^i \neq w_k^{i'} = j$ . Then every maximal chain in  $I$  has a unique edge with label  $(k, i, j)$ , and every edge label in  $I$  is greater than or equal to  $(k, i, j)$ .*

*Proof.* Condition (1) follows immediately from the definition of  $\lambda$ .

In Condition (2),  $\lambda(x \leq y) = (k, i, j)$  where  $k \leq n-1$  and  $i \leq s$ . Also,  $\lambda(\hat{0} \leq C_1) = (n-1, s+m, 0)$  for some  $m > 0$ . Therefore  $(n-1, s+m, 0) > (k, i, j)$ , which implies that this chain is not increasing.

We will now prove condition (3). Suppose for a contradiction that  $j > w_k^i$ . Now, the restriction of  $A(P, w^1, \dots, w^s)$  (respectively  $A(P, w^{1'}, \dots, w^{s'})$ ) to the subword  $w_1^i, \dots, w_n^i$  (respectively  $w_1^{i'}, \dots, w_n^{i'}$ ), is equal to the atom  $A(P, w^i)$  (respectively  $A(P, w^{i'})$ ) in  $\Pi_{n,2}$ . Therefore, by Condition (1), we have that  $j < w_k^i$ .

We will now prove condition (4). By the definition of  $A(P, w^1, \dots, w^s)$ , if  $k$  is the  $\alpha$ th largest element in a part  $L$ , then  $w_k^i$  is the  $\alpha$ th largest element in the  $i$ th label of  $L$ . The same is true of any element and any part, not just the element  $k$  and the part  $L$ . Suppose that, for a contradiction,  $j$  is contained in the  $i$ th label of  $L$ . Then,  $j < w_k^i$ , means that  $j$  is less than the  $\alpha$ th largest element in the  $i$ th label of  $L$ . Then there is an integer  $k - \epsilon$  ( $\epsilon$  is a positive integer) which is less than  $k$ , and is contained in the part  $L$ , which has the property that  $w_{k-\epsilon}^i = j$ . Thus,  $w_{k-\epsilon}^i \neq w_{k-\epsilon}^{i'}$ , which contradicts that  $(k, i, j)$  is the first triple such that  $w_k^i \neq w_k^{i'}$ .

We will now prove condition (5). Suppose for a contradiction that there is an edge with label  $(\bar{k}, \bar{i}, \bar{j}) < (k, i, j)$  in  $I$ . Suppose also that  $(\bar{k}, \bar{i}, \bar{j})$  is the lowest edge label in  $I$ . Suppose that  $(\bar{k}, \bar{i}) \neq (k, i)$ , and consider a maximal chain  $C$  in  $I$  that contains an edge with label  $(\bar{k}, \bar{i}, \bar{j})$ . By condition (3), and the fact that  $(\bar{k}, \bar{i}, \bar{j})$  is the lowest edge label in  $I$ , it must be true that higher edges in  $C$  with a label of the form  $(\bar{k}, \bar{i}, *)$  have lower values of  $*$ . Then we have a contradiction, since this implies that  $(k, i, j)$  is not the lowest index such that  $w_k^i \neq w_k^{i'} = j$ . Suppose now that  $(\bar{k}, \bar{i}) = (k, i)$ . Consider a chain  $C$  that contains an edge with the label  $(\bar{k}, \bar{i}, \bar{j})$ . Again, by condition (3), and the fact that  $(\bar{k}, \bar{i}, \bar{j})$  is the lowest label in  $I$ , we know that higher edges in  $C$  with a label of the form  $(\bar{k}, \bar{i}, *)$  have lower values of  $*$ . Therefore, the highest edge with a label of the form  $(\bar{k}, \bar{i}, *)$  in  $C$  is the edge  $(\bar{k}, \bar{i}, \bar{j})$ . Again, this implies the contradiction that  $(k, i, j)$  is not the earliest index such that  $w_k^i \neq w_k^{i'} = j$ . Therefore,  $(k, i, j)$  is the lowest edge label in  $I$ . By condition (4), this label occurs when the part that contains  $k$  is merged with the part whose  $i$ th label contains  $j$ , and therefore this edge label occurs only once in every maximal chain. It is clear that an edge with label  $(k, i, j)$  must occur once in each maximal chain.  $\square$

**Theorem 3.5.** *The edge labeling  $\lambda$  of  $\Pi_{n,s+1}$  is an EL-labeling.*

*Proof.* Consider intervals in  $\Pi_{n,s+1}$  of length 1. Such intervals contain exactly one edge, so that the restriction of  $\lambda$  to this interval is an EL-labeling. We assume by way of induction that the restriction of  $\lambda$  to any interval of length  $m$  for some  $m < n$  is an EL-labeling. We will now show that  $\lambda$  restricts to an EL-labeling on any interval of length  $m + 1$ .

Suppose  $I = [(P, w^1, \dots, w^s), (P', w^{1'}, \dots, w^{s'})]$  is an interval of length  $m + 1$ , and  $A(P, w^1, \dots, w^s) \neq A(P', w^{1'}, \dots, w^{s'})$ . Suppose that  $(k, i, j)$  is the lexicographically first element of  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$  such that  $w_k^i \neq w_k^{i'}$ , and  $w_k^{i'} = j$ . By Lemma 3.4 part (5), every chain in  $I$  has a unique edge with label  $(k, i, j)$ , and all edges have label greater than or equal to  $(k, i, j)$ . This implies that there is a unique atom  $a$  in  $I$  such that  $\lambda((P, w^1, \dots, w^s) \leq a) = (k, i, j)$ , which, by Lemma 3.4 part (4), is obtained from  $(P, w^1, \dots, w^s)$  by merging the part that contains  $k$  with the part whose  $i$ th label contains  $j$ . By induction, the interval  $[a, (P', w^{1'}, \dots, w^{s'})]$  contains a unique increasing maximal chain which lexicographically precedes other chains. The concatenation of this chain with  $(P, w^1, \dots, w^s) \leq a$  gives a chain  $C'$  which is the increasing because all edge labels in  $[a, (P', w^{1'}, \dots, w^{s'})]$  must be greater than  $(k, i, j)$ . By induction,  $C'$  is the unique increasing maximal chain in  $I$  which contains  $(P, w^1, \dots, w^s) \leq a$ , and it lexicographically precedes other chains containing  $(P, w^1, \dots, w^s) \leq a$ . Since all chains in  $[(P, w^1, \dots, w^s), (P', w^{1'}, \dots, w^{s'})]$  must contain an edge with label  $(k, i, j)$ , and chains that don't contain  $(P, w^1, \dots, w^s) \leq a$  have lowest edge label greater than  $(k, i, j)$ ,  $C'$  is the unique increasing maximal chain in  $I$  which lexicographically precedes all other chains.

Suppose  $[(P, w^1, \dots, w^s), (P', w^{1'}, \dots, w^{s'})]$  is an interval of length  $m + 1$ , and  $A(P, w^1, \dots, w^s) = A(P', w^{1'}, \dots, w^{s'})$ . Recall that, by Proposition 3.3,

$$[(P, w^1, \dots, w^s), (P', w^{1'}, \dots, w^{s'})] \cong [P, P'],$$

and denote this map by  $\phi$ . Observe also that  $\lambda(x \leq y) = (n, \chi(\phi(x) \leq \phi(y)), 0)$  for all  $x \leq y \in [(P, w^1, \dots, w^s), (P', w^{1'}, \dots, w^{s'})]$ , where  $\chi$  is the edge-labeling given in Theorem 3.2. Hence there

is a unique increasing maximal chain that lexicographically precedes other chains in this interval.

Suppose  $I = [\hat{0}, (P, w^1, \dots, w^s)]$  is an interval of length  $m + 1$ . Then, by the inductive hypothesis, the interval  $[A(P, w^1, \dots, w^s), (P, w^1, \dots, w^s)]$  contains a unique increasing maximal chain that lexicographically precedes other chains in this interval. The concatenation of this chain with  $\hat{0} \triangleleft A(P, w^1, \dots, w^s)$ , denoted  $C'$ , is increasing, since by Lemma 3.4 part (1) all elements in the interval  $[A(P, w^1, \dots, w^s), (P, w^1, \dots, w^s)]$  have the same atom, and so all edge labels are of the form  $(n, *, *)$ . Therefore, by induction,  $C'$  is the unique increasing chain which lexicographically precedes chains in  $I$  that contain  $\hat{0} \triangleleft A(P, w^1, \dots, w^s)$ . Since  $A(P, w^1, \dots, w^s)$  is defined as the earliest atom that is less than  $(P, w^1, \dots, w^s)$ , it is clear that  $C'$  lexicographically precedes all maximal chains in  $I$ . Any chain in  $I$  that does not contain  $\hat{0} \triangleleft A(P, w^1, \dots, w^s)$  contains a descent, since they contain an edge  $x \triangleleft y$  in which  $A(x) \neq A(y)$ , so by Lemma 3.4 part(2), they are not increasing.  $\square$

We provide a proof of the following Theorem, which is immediately implied by the main results of [9]:

**Theorem 3.6** (Sagan [9]). *For any  $n$  and  $s$ , the order complex  $\Delta(\Pi_{n,s+1})$  is homotopy equivalent to a wedge of  $(n - 2)$ -dimensional spheres.*

*Proof.* By Theorem 3.5, the poset  $\Pi_{n,s+1}$  admits an EL-labeling, so that by Theorem 2.3, the order complex  $\Delta(\Pi_{n,s+1})$  is shellable. By Proposition 2.2, this implies that  $\Delta(\overline{\Pi}_{n,s+1})$  is shellable, and homotopy equivalent to a wedge of spheres. The spheres are of dimension  $(n - 2)$  since  $\overline{\Pi}_{n,s+1}$  is pure, and its maximal chains contain  $n - 1$  elements.  $\square$

#### 4. THE NUMBER OF SPHERES IN $\Pi_{n,s+1}$

Recall Theorem 2.3, which states that the number of spheres is equal to the number of decreasing maximal chains in  $\Pi_{n,s+1}$ . In this section, we characterise the decreasing maximal chains of  $\Pi_{n,s+1}$  in order to count them. In the case that  $s = 1$ , we show that the number of decreasing maximal chains is equal to the number of complete non-ambiguous trees with  $n$  internal vertices, which are defined in [1].

**Proposition 4.1.** *If  $C = C_0 \triangleleft C_1 \triangleleft \dots \triangleleft C_n$  is a decreasing maximal chain in  $\Pi_{n,s+1}$ , then  $A(C_i) \neq A(C_{i+1})$  for all  $i \in [n - 1]$ .*

*Proof.* Now  $\lambda(C_0 \triangleleft C_1) = (n - 1, s + j, 0)$  for some  $j \geq 1$ . By the definition of  $\lambda$ , if  $A(C_i) = A(C_{i+1})$  for some  $i \in [n - 1]$ , then  $\lambda(C_i \triangleleft C_{i+1}) = (n, *, *)$ , which implies the contradiction that  $\lambda(C)$  is not increasing. Therefore, must have that  $A(C_i) \neq A(C_{i+1})$  for all  $i \in [n - 1]$ .  $\square$

**Example 4.2.** *The chain*

$$\begin{aligned} C = \hat{0} \triangleleft (\{1\}\{2\}\{3\}\{4\}\{5\}, \{5\}\{4\}\{1\}\{3\}\{2\}) &\triangleleft (\{1\}\{2, 4\}\{3\}\{5\}, \{5\}\{34\}\{1\}\{2\}) \\ &\triangleleft (\{1\}\{2, 3, 4\}\{5\}, \{5\}\{1, 3, 4\}\{2\}) \triangleleft (\{1, 5\}\{1, 3, 4\}, \{25\}\{134\}) \\ &\triangleleft (\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}) \end{aligned}$$



is maximal decreasing in the poset  $\Pi_{5,2}$ . The decreasing sequence of edge labels for this chain is  $\lambda(C) = (4, 5 + 116, 0)(2, 1, 3)(2, 1, 1)(1, 1, 2)(1, 1, 1)$ . This example also demonstrates the condition on decreasing chains given in Propositions 4.3 and 4.5.

**Proposition 4.3.** Suppose that  $(P, w^1, \dots, w^s) \triangleleft (P', w^{1'}, \dots, w^{s'})$  is an edge in a decreasing maximal chain of  $\Pi_{n,s+1}$ , in which  $\lambda((P, w^1, \dots, w^s) \triangleleft (P', w^{1'}, \dots, w^{s'})) = (k, i, j)$ . Then  $P'$  is of the form  $\{1\}\{2\} \dots \{k-1\}I_k \dots I_m$ , where  $I_k$  is a non singleton set. The partition  $P$  is of the form  $\{1\}\{2\} \dots \{k-1\}I_\alpha I_{k+1} \dots I_\beta \dots I_m$  where  $I_\alpha$  and  $I_\beta$  are a partition of  $I_k$  (note that  $I_\beta$  may be to the left of  $I_{k+1}$  or to the right of  $I_m$ ). For all  $h < i$ , the  $h$ th label of  $I_\alpha$  contains the minimal element of the  $h$ th label of  $I_k$ , and the  $i$ th label of  $I_\beta$  contains the minimal element of the  $i$ th label of  $I_k$ .

*Proof.* First, we will show that  $P'$  is of the form  $\{1\} \dots \{k-1\}I_k \dots I_m$ , where  $I_k$  is a non-singleton set.

Assume that  $(P, w^1, \dots, w^s)$  is an atom of  $\Pi_{n,s+1}$ , so that  $P = \{1\} \dots \{n\}$ . Then  $P' = \{1\} \dots \{k-1\}\{kf\} \dots \{n\}$ , where  $\{f\} \in \{k+1, \dots, n\}$ . This is true since  $w_k^i \neq w_k^{i'}$  implies that  $w_f^i \neq w_f^{i'}$ , and so  $f < k$  contradicts that  $(k, i, j)$  is the first triple such that  $w_k^i \neq w_k^{i'}$ .

Assume by induction that  $P'$  is of this form if  $P'$  has  $m$  parts where  $n-1 \leq m \leq 2$ . We will show that  $P'$  has this form when it has  $m+1$  parts. If  $C_{n-m-1} \triangleleft (P, w^1, \dots, w^s) \triangleleft (P', w^{1'}, \dots, w^{s'})$  is contained in a maximal decreasing chain, then  $\lambda(C_{n-m-1} \triangleleft (P, w^1, \dots, w^s)) \geq (k, i, j)$ , and by induction,  $P = \{1\} \dots \{k-1\}I_k \dots I_{m+1}$ , where  $I_k$  contains  $k$  and is a non-singleton set exactly when  $\lambda(C_{n-m-1} \triangleleft (P, w^1, \dots, w^s)) = (k, *, *)$ . By Lemma 3.4 Part (4),  $P'$  is obtained from  $P$  by merging the part that contains  $k$  with the part whose  $i$ th label contains  $j$ . Now the part whose  $i$ th label contains  $j$  cannot be any of  $\{1\}, \dots, \{k-1\}$ , as this would imply that  $(k, i, j)$  is not the first triple such that  $w_k^i \neq w_k^{i'}$ . Hence  $P'$  is of the form  $\{1\} \dots \{k-1\}I_k \dots I_m$  where  $I_k$  is a non-singleton set.

By Lemma 3.4 Part (4),  $(P', w^{1'}, \dots, w^{s'})$  is obtained from  $(P, w^1, \dots, w^s)$  by merging the part whose  $i$ th label contains  $j$  (the set  $I_\beta$ ) with the part that contains  $k$  (the set  $I_\alpha$ ). Since  $k$  is the least integer in the set  $I_\alpha$ , it is only possible that  $(k, i, j)$  is the first triple such that  $w_k^i \neq w_k^{i'} = j$  if for every  $h < i$ , the minimal element of the  $h$ th label of  $I_k$  is contained in the  $h$ th label of  $I_\alpha$ , and the minimal element of the  $i$ th label of  $I_k$  (the number  $j$ ) is contained in the  $i$ th label of  $I_\beta$ . □

**Example 4.4.** The chain  $\hat{0} = C_0 \triangleleft C_1 \triangleleft \dots \triangleleft C_5$  in  $\Pi_{5,3}$  is maximal decreasing.

$$\begin{aligned} C_5 &= ([5], [5], [5]) \\ C_4 &= (\{1, 4, 5\}\{2, 3\}, \{2, 3, 5\}\{1, 4\}, \{3, 4, 5\}\{1, 2\}) \\ C_3 &= (\{1\}\{2, 3\}\{4, 5\}, \{2\}\{1, 4\}\{3, 5\}, \{4\}\{1, 2\}\{3, 5\}) \\ C_2 &= (\{1\}\{2\}\{3\}\{4, 5\}, \{2\}\{4\}\{1\}\{3, 5\}, \{4\}\{1\}\{2\}\{3, 5\}) \\ C_1 &= (\{1\}\{2\}\{3\}\{4\}\{5\}, \{2\}\{4\}\{1\}\{3\}\{5\}, \{4\}\{1\}\{2\}\{5\}\{3\}) \\ C_0 &= \hat{0}. \end{aligned}$$

Then

$$\begin{aligned}\lambda(C_0 \triangleleft C_1) &= (4, 2 + m, 0) \text{ for some } m, \quad \lambda(C_1 \triangleleft C_2) = (4, 2, 3), \quad \lambda(C_2 \triangleleft C_3) = (2, 1, 1) \\ \lambda(C_3 \triangleleft C_4) &= (1, 2, 3), \quad \lambda(C_4 \triangleleft C_5) = (1, 1, 1),\end{aligned}$$

and

$$\begin{aligned}A(C_1) &= (24135)(41253), \quad A(C_2) = (24135)(41235), \quad A(C_3) = (21435)(41235) \\ A(C_4) &= (21435)(31245), \quad A(C_5) = (12345)(12345)\end{aligned}$$

where we have added brackets into the expressions for the atoms for clarity.

**Proposition 4.5.** *Suppose  $C_d \triangleleft \dots \triangleleft C_n$  is the upper portion of a maximal decreasing chain in  $\Pi_{n,s+1}$ . Suppose that  $C_d = (P, w^1, \dots, w^s)$ , where  $P = \{1\} \dots \{k-1\} I_k \dots I_{n-d+1}$  and  $I_k$  is a non-singleton set. Suppose that  $C_{d-1} \triangleleft C_d$  is of the following form:*

- *If  $\lambda(C_d \triangleleft C_{d+1}) = (k, i, j)$  for some  $i$  and  $j$ , then the partition in  $C_{d-1}$  is of the form  $\{1\} \dots \{k-1\} I_\alpha I_{k+1} \dots I_\beta \dots I_{n-d+1}$ , where  $I_\alpha$  and  $I_\beta$  are a partition of  $I_k$  (it is possible that  $I_\beta$  is to the left of  $I_{k+1}$  or to the right of  $I_{n-d+1}$ ) and  $k \in I_\alpha$ . Also, for some  $i' \geq i$ , the  $i'$ th label of  $I_\beta$  contains  $w_k^i$  (the minimal element of the  $i$ th label of  $I_k$ ), and for all  $h < i'$ , the  $h$ th label of  $I_\alpha$  contains  $w_k^h$ .*
- *If  $\lambda(C_d \triangleleft C_{d+1}) = (k', i, j)$  for some  $k' < k$ , or if  $d = n$ , then  $C_{d-1}$  is obtained by partitioning  $I_k$  into two non-empty sets  $I_\alpha$  and  $I_\beta$  so that  $k \in I_\alpha$ , and for some index  $i'$  the  $i'$ th label of  $I_\beta$  contains  $w_k^i$ .*

then  $C_{d-1} \triangleleft \dots \triangleleft C_n$  is the upper portion of a maximal decreasing chain in  $\Pi_{n,s+1}$ .

*Proof.* It is immediate from the definition of  $\lambda$  that if  $C_{d-1}$  is obtained using these methods, then the chain  $C_{d-1} \triangleleft \dots \triangleleft C_n$  is decreasing and  $A(C_i) \neq A(C_j)$  when  $i \neq j$ . The element  $C_{d-1}$  is of the same form as  $C_d$ , and so we can continue to extend this chain using this process, and any resulting chain will also have the property that it is decreasing. After performing  $d-1$  iterations and concatenating the chain with  $\hat{0}$ , the resulting chain will be decreasing maximal.  $\square$

Let  $\mathcal{B}_n^{s+1}$  denote the set of maximal decreasing chains in  $\Pi_{n,s+1}$ . For  $n > 1$ , let  $\mathcal{B}_n^{s+1,i}$  denote the set of maximal decreasing chains in  $\Pi_{n,s+1}$  whose highest edge has label  $(k, i, j)$  for some  $k, j$ .

**Proposition 4.6.** *Any given  $C \in \mathcal{B}_n^{s+1,i}$  is defined uniquely by the following data. Here we index the  $s+1$  partitions in any element of  $\Pi_{n,s+1}$  by the integers 0 to  $s$ :*

- (1) *The element  $C_{n-1}$  in  $C$ . The element  $C_{n-1}$  can be described as a partition into a left and right part, in which the left part is of size  $\alpha$ , of each set in  $C_n = ([n], \dots, [n])$ . For all  $h < i$ , the  $h$ th element of  $([n], \dots, [n])$  is partitioned so that the left part contains 1, and the  $i$ th element in  $([n], \dots, [n])$  is partitioned so that 1 is contained in the right part.*
- (2) *An element  $C_L \in \mathcal{B}_\alpha^{s+1,i'}$  for some  $2 \leq \alpha \leq n-1$  and some  $i' \geq i$ , or by an element  $C_L \in \mathcal{B}_\alpha^{s+1}$  where  $\alpha = 1$ .*
- (3) *An element  $C_R \in \mathcal{B}_{n-\alpha}^{s+1}$ .*

*Proof.* To prove this we first use Propositions 4.1 4.3 and 4.5 to characterise decreasing maximal chains in  $\Pi_{n,s+1}$ . Together, these propositions imply that any given decreasing maximal chain in  $\Pi_{n,s+1}$  can be obtained by starting with the element  $C_n = ([n], \dots, [n])$  and repeatedly applying steps (1) or (2) of Proposition 4.5. Also, by Proposition 4.5, by repeatedly applying steps (1) or (2), one will obtain a maximal decreasing chain. Therefore, we can characterise maximal decreasing chains of  $\Pi_{n,s+1}$  by considering the ways we can apply steps (1) or (2). We will first argue that (1)-(3) partially define elements of  $\mathcal{B}_n^{s+1,i}$ . We will then argue that (1)-(3) define these elements uniquely.

It is clear that point (1) partially describes elements in  $\mathcal{B}_n^{s+1,i}$ .

Suppose  $C_{n-1} = (P, w^1, \dots, w^s)$ , where  $P = I_1 I_2$ , and suppose that  $|I_1| = \alpha$ . Consider all of the parts and labels in elements in  $C$  that are subsets of  $I_1$  and its labels. Form a new chain  $C_L \in \mathcal{B}_\alpha^{s+1}$  as follows. From each element  $C_p \in C$  where  $p \leq n-1$  remove all parts that are subsets of  $I_2$  and remove their corresponding labels. Next, relabel the elements in each part and label so that the  $\alpha$ th greatest element in  $I_1$  is replaced with  $\alpha$ , and the  $\alpha$ th greatest element in the labels of  $I_1$  are replaced with  $\alpha$ . Now, some of the elements in  $\Pi_{\alpha,s+1}$  we obtained will be duplicates. If we remove the duplicates, then the set of elements form a maximal decreasing chain in  $\mathcal{B}_\alpha^{s+1,i'}$  for some  $i' \geq i$ , or in  $\mathcal{B}_1$  (see Example 4.7). This is the chain  $C_L$  that partially defines  $C$  in point (2).

Similar to the method used to obtain  $C_L$ , we obtain  $C_R$  by removing parts that contain  $I_1$  and their corresponding labels, and we then renumber the elements in each part and label (see Example 4.7). Thus,  $C$  is also defined in part by an element  $C_R \in \mathcal{B}_{n-\alpha}^{s+1}$ , as described in point (3).

We have shown that points (1)-(3) are properties of any chain  $C \in \mathcal{B}_n^{s+1,i}$ . Now given the data of (1)-(3) for a chain  $C$ , we will argue that we can construct  $C$  uniquely from this data. We prove this by induction. Assume that we are able to deduce the elements  $C_n, C_{n-1}, \dots, C_d$  uniquely for some  $d \leq n-1$ . Recall that  $C_{n-1}$  has partition  $I_1 I_2$ . Assume that  $C_d = (P, w^1, \dots, w^s)$ , where  $P = \{1\}, \dots, \{k-1\} I_k, \dots, I$ , where  $I_k$  is a non singleton set. Now the part  $I_k$  is contained in either  $C_L$  or  $C_R$ , and this can be deduced by considering whether the elements in  $I_k$  are in  $I_1$  or  $I_2$  of  $C_{n-1}$ . We partition the set  $I_k$  and its corresponding labels in the same manner that they are partitioned in the corresponding chain  $C_L$  or  $C_R$ . This determines  $C_{d-1}$  uniquely, and therefore, by induction, the entire chain  $C$  can be determined uniquely.  $\square$

**Example 4.7.** Suppose  $C$  is the maximal decreasing chain from Example 4.4. We rewrite it here for convenience.

$$\begin{aligned}
C_5 &= ([5], [5], [5]) \\
C_4 &= (\{1, 4, 5\}\{2, 3\}, \{2, 3, 5\}\{1, 4\}, \{3, 4, 5\}\{1, 2\}) \\
C_3 &= (\{1\}\{2, 3\}\{4, 5\}, \{2\}\{1, 4\}\{3, 5\}, \{4\}\{1, 2\}\{3, 5\}) \\
C_2 &= (\{1\}\{2\}\{3\}\{4, 5\}, \{2\}\{4\}\{1\}\{3, 5\}, \{4\}\{1\}\{2\}\{3, 5\}) \\
C_1 &= (\{1\}\{2\}\{3\}\{4\}\{5\}, \{2\}\{4\}\{1\}\{3\}\{5\}, \{4\}\{1\}\{2\}\{5\}\{3\}) \\
C_0 &= \hat{0}.
\end{aligned}$$

Here  $I_1 = \{1, 4, 5\}$  and  $I_2 = \{2, 3\}$ . After removing subsets of  $I_2$  and their corresponding labels,  $C_3$ , for example, becomes

$$(\{1\}\{4, 5\}, \{2\}\{3, 5\}, \{4\}\{3, 5\}),$$

and under the relabeling this becomes

$$(\{1\}\{2, 3\}, \{1\}\{2, 3\}, \{2\}\{1, 3\}).$$

Then  $C_L$  is the chain

$$\hat{0} < (\{1\}\{2\}\{3\}, \{1\}\{2\}\{3\}, \{2\}\{3\}\{1\}) < (\{1\}\{2, 3\}, \{1\}\{2, 3\}, \{2\}\{1, 3\}) < ([3], [3], [3]),$$

and  $C_R$  is the chain

$$\hat{0} < (\{1\}\{2\}, \{2\}\{1\}, \{1\}\{2\}) < ([2], [2], [2]).$$

Using Proposition 4.6 we are able to count the number of maximal decreasing chains in  $\Pi_{n,s+1}$ . We do not find an exact solution for any  $n$  and  $s$ , but we do find a recurrence relation which simplifies considerably when  $s = 1$ . In the  $s = 1$  case, a solution to the recurrence is not known, but we show that the recurrence is the same as the recurrence found in [1] which counts the number of complete non-ambiguous trees.

**Theorem 4.8.** *For any  $n \geq 2$ , any  $s \geq 1$ , and any  $1 \leq i \leq s$  the following recursion holds*

$$(4.1) \quad |\mathcal{B}_n^{s+1,i}| = \sum_{1 \leq \alpha \leq n-1} \sum_{i' \geq i} |\mathcal{B}_\alpha^{s+1,i'}| |\mathcal{B}_{n-\alpha}^{s+1}| \binom{n-1}{\alpha-1}^i \binom{n-1}{\alpha} \binom{n}{\alpha}^{s-i},$$

where,  $|\mathcal{B}_1^{s+1}| = 1$  for all  $s$ . When  $\alpha = 1$  in Equation 4.1, we let  $\mathcal{B}_\alpha^{s+1,i'} = \mathcal{B}_\alpha^{s+1}$ , and there is no summation over  $i'$ .

*Proof.* We have characterised decreasing maximal chains in Proposition 4.6.

The index  $\alpha$  in Equation 4.1 is the size of the left parts in point (1). If the left parts are of size  $\alpha$ , then there are  $\binom{n-1}{\alpha-1}^i$  choices for the partitions the parts indexed from 0 to  $i-1$ , since the left parts all contain 1, leaving a choice of  $\alpha-1$  from  $n-1$  remaining elements in the left parts. There are  $\binom{n-1}{\alpha}$  ways to choose elements for the left part of the  $i$ th label in  $C_n = ([n], \dots, [n])$ , and there are  $\binom{n}{\alpha}^{s-i}$  ways to partition the remaining labels in  $C_n = ([n], \dots, [n])$ . These choices contribute the  $\binom{n-1}{\alpha-1}^i \binom{n-1}{\alpha} \binom{n}{\alpha}^{s-i}$  terms in Equation 4.1.

For conditions (2) and (3), the number of choices for  $C_L$  and  $C_R$  for any particular  $\alpha$  and label  $i' \geq i$  of the  $C_L$ , is given by  $|\mathcal{B}_\alpha^{s+1,i'}| |\mathcal{B}_{n-\alpha}^{s+1}|$ .

□

We will now show that the set of maximal decreasing chains in  $\Pi_{n,2}$  is in one to one correspondence with the set of complete non-ambiguous trees with  $n$  leaves, which are defined and studied in [1]. Theorem 4.8 shows that when  $s = 1$ , the number of maximal decreasing chains in  $\Pi_{n,2}$  satisfies the following recurrence:

$$(4.2) \quad |\mathcal{B}_n^2| = \sum_{1 \leq \alpha \leq n-1} \binom{n-1}{\alpha-1} \binom{n-1}{\alpha} |\mathcal{B}_\alpha^2| |\mathcal{B}_{n-\alpha}^2|,$$

with  $\mathcal{B}_1^2 = 1$ .

In [1], the number of non-ambiguous trees with  $n$  internal vertices is denoted by  $b_n$  and is shown to satisfy the recurrence

$$(4.3) \quad b_{n+1} = \sum_{i+j=n} \binom{n+1}{i} \binom{n+1}{j} b_i b_j,$$

for  $n \geq 0$ , with  $b_0 = 1$ .

**Corollary 4.9.** *The order complex  $\Delta(\Pi_{n,2})$  is homotopy equivalent to a wedge of  $b_{n-1}$   $(n-2)$ -spheres, where  $b_n$  is the number of complete non-ambiguous trees with  $n$  internal vertices.*

*Proof.* By equations 4.2 and 4.3 it follows that  $b_{n-1} = |\mathcal{B}_n^2|$  for all  $n \geq 1$ . □

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